I. Fundamental Theorem of Galois Theory :-

Statement :- Let K be a finite normal Extension of a field F and characteristic root is Zero. Then $G(K,F)$ be Galois Group of K over F. Then follow. Correspondance b/w $E \longleftrightarrow G(K,E)$ where E is a subfield of K and $G(K,E)$ is Subgroup of $G(K,F)$ and has one-one Correspondance b/w family of Subfield of K and family of Subgroup of $G(K,F)$

Then five Condition are arise :

i) $E = K_{G(K,E)}$

ii) $H = G[K, K_H)$

iii) $[K:E] = O[G(K,E)]$

& $[E:F]$ = Index of $G(K,E)$ in $G(K,F)$

iv) 9f K

v) $\dfrac{G(K,F)}{G(K,E)} \cong G(E,F)$

Proof: i) Let K be finite normal extension of F & $F \subseteq E \subseteq K$

Then K be finite normal ext. of E Then fixed field under $G(K,E)$ is E itself.

Then $E = K_{G(K,E)}$

ii) By Def.

$K_H = \{ x \in K : \sigma(x) = x \ \forall \sigma \in H \}$

$\sigma \in H, \sigma(x) = x \ \forall x \in K_H$

$\therefore \sigma$ is K_H automorphism of K

$\therefore \sigma \in G(K, K_H)$

$\Rightarrow H \subseteq G(K, K_H)$

Also $O(H) = [K : K_H]$ K_H is fixed field ext. of H

$\& \ [K : K_H] = O[G(K : K_H)]$

$\& \ O(H) = O[G(K : K_H)]$

$\therefore \ H = G(K, K_H)$

iii) Let K be finite normal Ext. of E

$$[K : E] = O[G(K : E)]$$

$$O[G(K : F)] = [K : F] = [K : E][E : F]$$

$$O[G(K : F)] = O[G(K : E)] \, [E : F]$$

$$\frac{O[G(K : F)]}{O[G(K : E)]} = [E : F]$$

$= \text{index of } G(K, E) \text{ in } G[K, F]$

iv) E be finite Normal Ext. of F.
Then $a \in E$ & $p(x)$ be minimal polynomial of a over F.

Then $p(x)$ is irreducible Polynomial over F having a root 'a' in E.

So $p(x)$ has all root in E.

Thus Every Conjugate of 'a' over F in K is again in E.

$\sigma \in G(K, F)$

$\sigma(a)$ is Conjugate of 'a'.

$\sigma(a) \in E$

Let $\eta \in G(K, E)$

$\eta[\sigma(a)] = \sigma(a)$

So $(\sigma^{-1}\eta\sigma)(a) = a$

$\therefore \sigma^{-1}\eta\sigma \in G(K,E)$

for $\sigma \in G(K,F)$ & $\eta \in G(K,E)$

Hence $G(K,E)$ is Normal Subgroup of $G(K,F)$

Converse: Let $G(K,E)$ is normal subgroup of $G(K,F)$

Let $p(x)$ is irreducible polynomial over F having root 'a' in E.

$\sigma : K \to K$

$\sigma(a) = b$ & $\sigma(\alpha) = \alpha$, $\forall \alpha \in F$

$\sigma \in G(K,F)$

Since $G(K,E)$ is Normal subgrp of $G(K,F)$

$\sigma^{-1}\eta\sigma(a) \in G(K,E)$ $\forall$ $\eta \in G(K,E)$

$\eta\sigma(a) = \sigma(a)$

$\sigma(a) \in K_{G(K,E)} = E$

i.e. $b \in E$

Hence $p(x)$ has all roots in K.

Hence E is Normal Ext. of F.

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v) Let E be Normal Ext. of F

T.P $\frac{G(K,F)}{G(K,E)} \cong G(E,F)$

Define a mapping

$f: G(K,F) \to G(E,F)$

$f(\sigma) = \sigma_E \quad \forall \sigma \in G(K,F)$

$\sigma, \eta \in G(K,F)$

$(\sigma\eta)_E = \sigma_E \eta_E$

$f(\sigma\eta) = (\sigma\eta)_E$

$= \sigma_E \eta_E$

$f(\sigma\eta) = f(\sigma) f(\eta)$

$\therefore$ f is homomorphism.

By fund. Thm of homomorphism,

$G(E,F) \cong \frac{G(K,F)}{\mathrm{Ker}\, f}$ — (1)

Let $\sigma \in G(K,F)$

$\sigma \in \mathrm{Ker}\, f$

σ_E is identity on E

$\sigma(x) = x \quad \forall x \in E$

$\therefore \sigma \in G(K,E)$

$\mathrm{Ker}\, f = G(K,E)$

Hence

$G(E,F) \cong \frac{G(K,F)}{G(K,E)}$

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